The Simplex Method

The simplex method is an algebraic method for solving linear programming problems. It is not difficult to implement (once you get used to the vocabulary!) but it is difficult to explain WHY the method works. The advantages of the simplex method are

1) It can be used to tackle problems in which there are more than two decision variables (not possible with graphical methods).
2) It is a method that can be programmed on a computer fairly easily.

A disadvantage of the simplex method is that it can only be applied to certain types of linear programming problems. The types of problems for which the simplex method are suitable are ones that can be expressed in the following STANDARD FORM:

- The constraints must include the non-negativity constraints for all the decision variables.
- The other constraints must be expressed in the form (linear expression) $\leq$ No. where the No. on the right hand side must be positive.
- The objective must be to MAXIMISE some linear expression.

While it is always possible to meet the third of these conditions, it is not always possible to meet the first or second one. For example, if a linear programming problem specifies that a function $P = 3x - 4y$ is to be minimised, we may solve the problem by maximising $-P = -3x + 4y$ instead. However, if the non-negativity constraints are not present, there is nothing we can do about it. Similarly, if one of the constraints requires that $2x + 5y \geq 2$, we cannot transform this to an inequality of the required form. Multiplying through by -1 gives $-2x - 5y \leq -2$, but standard form requires the constant on the right hand side to be POSITIVE.

The following example illustrates HOW to implement the simplex method without attempting to explain WHY. It is this type of question that will be presented as an exam at the end of semester 1. For the time being, view the simplex method as a "black box" into which you put a linear programming problem and out of which comes the solution!

Illustrative Example

Consider the following linear programming problem:

Maximise $P = 3x + 5y$ subject to the constraints:

$$12x + 5y \leq 500$$
$$x + 2y \leq 100$$
$$4x + 5y \leq 275$$
$$\text{and } x \geq 0,\ y \geq 0$$

The steps in the simplex method, as we are going to implement it, are as follows:

1) First note that the problem is stated in standard form. Were this not the case we would need to express it in standard form. If this were not possible we would be stuck!
2) Convert the inequalities in the problem (the constraints) into equations by introducing SLACK VARIABLES. Letters such as s, t and u (or w_1, w_2, w_3) normally denote these variables. A slack variable represents the difference between the linear expression on the left hand side of an inequality and its maximum value. Notice that slack variables CANNOT be negative if the inequalities are to be satisfied.

   The inequalities in the example are replaced by the equations:

   $12x + 5y + s = 500$ (1)
   $x + 2y + t = 100$ (2)
   $4x + 5y + u = 275$ (3)

   where it is required that $x \geq 0$, $y \geq 0$, $s \geq 0$, $t \geq 0$ and $u \geq 0$.

   The objective function is also rewritten so that it has a 0 on the right hand side. In the example, the objective function $P = 3x + 5y$ is rearranged to

   $P - 3x - 5y = 0$ (4)

3) Construct the first SIMPLEX TABLEAU. For the example it is:

   Tableau 1

Basic Variables	X	Y	S	T	U	Values
S	1	5	1	0	0	500
T	1	2	1	0	1	100
U	4	5	0	1	2	275
P	-3	-5	0	0	0	0

   The tableau is constructed from equations (1) to (4). The numerical entries in the table are the coefficients of the variables that appear at the top of the columns. The “Basic Variables” are initially s, t and u. The simplex tableaux that we are going to construct will all contain the columns

1
0
0

0
1
0

0
0
1

in them somewhere (ignoring the 0 entries in the bottom row). It is these columns that allow the basic variables to be identified. Looking at equations (1) to (4), notice that if we put x = 0 and y = 0, the values taken by the basic variables are immediately apparent – more on this later. The variables x and y are said to be NON-BASIC variables at this stage.

Look at the bottom row of the tableau – notice that there are negative entries in this row, namely -3 and -5. These negative entries indicate that it is necessary to construct another simplex tableau. The simplex method terminates and the solution to the linear programming problem can be found when there are no negative entries in the bottom row. Steps 4) – 6) are concerned with constructing the second simplex tableau.

4) Identify the PIVOT. The PIVOTAL COLUMN is identified by looking at the bottom row of the table. The pivotal column is the column with the MOST NEGATIVE entry in this row. In our example, the pivotal column is the one with -5 at the bottom (and y at the top). The PIVOTAL ROW is obtained by dividing the numbers in the "Values" column by the corresponding numbers in the pivotal column. Add the results of these calculations into a new column on tableau 1 headed θ. This is shown below:

Basic Variables	X	Y	S	T	U	Values	θ
S	1	5	1	0	0	500	500/5=100
T	1	2	0	1	0	100	100/2=50
U	4	5	0	0	1	275	275/5=55
P	-3	-5	0	0	0	0	

The pivotal row is then simply the row containing the smallest θ value – in this case, it is the row with 50 in the θ column (and t at the start). The PIVOT is the number that lies at the intersection of the pivotal column and the pivotal row. In our example, the pivot is the number 2 that lies in the column headed y and in the row with t at its front.

Basic Variables	X	Y	S	T	U	Values	θ
S	1	5	1	0	0	500	500/5=100
T	1	**2**	0	1	0	100	100/2=50
U	4	5	0	0	1	275	275/5=55
P	-3	-5	0	0	0	0	

Steps 5) and 6) describe the calculations that must be done to construct tableau 2. When these calculations are complete, the part of the y column that currently contains 5, 2, 5 will contain 0, 1, 0 and the column headed t will no longer contain

0, 1, 0, thus the variable y will become a basic variable and the variable t will become a non-basic variable. In the language of the simplex method, we say that y is ENTERING THE BASIS and t is LEAVING THE BASIS. More briefly, y is said to be the ENTERING variable and t is said to be the LEAVING variable.

5) Divide all of the numbers in the pivotal row by the pivot. This gives:

Basic Variables	X	Y	S	T	U	Values
S	1	5	1	0	0	500
T	$^1/_2$	1	0	$^1/_2$	0	50
U	4	5	0	0	1	275
P	-3	-5	0	0	0	0

6) Now subtract multiples of the pivotal row from each of the other rows so that the pivotal column contains 0s in all rows APART from the pivotal row (which contains a 1). The calculations that are required are indicated to the right of the table below, and tableau 2 is given below that:

Basic Variables	X	Y	S	T	U	Values	
S	1	5	1	0	0	500	R_1-$5R_2$
T	$^1/_2$	1	0	$^1/_2$	0	50	Pivot Row
U	4	5	0	0	1	275	R_3-$5R_2$
P	-3	-5	0	0	0	0	R_4+$5R_2$*

* This calculation can be thought of as R_4 – ($-5R_2$) if you prefer (the row calculations really are all subtractions then).

Tableau 2

Basic Variables	X	Y	S	T	U	Values
S	$^{19}/_2$	0	1	$^{-5}/_2$	0	250
T	$^1/_2$	1	0	$^1/_2$	0	50
U	$^3/_2$	0	0	$^{-5}/_2$	1	25
P	$^{-1}/_2$	0	0	$^5/_2$	0	250

Having constructed tableau 2, look at its bottom row. Notice that there IS a negative entry in this row and we therefore conclude that it is necessary to construct a third simplex tableau. Steps 7) to 9) are concerned with constructing the third simplex tableau.

7) Identify the PIVOT in the same way as before. Look for the most negative entry in the bottom row of the table to identify the pivotal column. In our example, the

pivotal column is the one with $^{-1}/_2$ at the bottom (and x at the top). The pivotal row is obtained by dividing the numbers in the "Values" column by the corresponding numbers in the pivotal column. The results of these calculations are entered into a new column on tableau 2 headed θ. This is shown below:

Basic Variables	X	Y	S	T	U	Values	θ
S	$^{19}/_2$	0	1	$^{-5}/_2$	0	250	$250/^{19}/_2=^{500}/_{19}$
T	$^{1}/_2$	1	0	$^{1}/_2$	0	50	$50/^{1}/_2=100$
U	$^{3}/_2$	0	0	$^{-5}/_2$	1	25	$25/^{3}/_2=^{50}/_3$
P	$^{-1}/_2$	0	0	$^{5}/_2$	0	250	

The pivotal row is the row containing the smallest θ value – in this case, it is the row with $^{50}/_3$ in the θ column (and u at the start). The PIVOT is thus $^{3}/_2$ and it lies in the column headed x and the row with a u at the beginning. We may deduce that x will be entering the basis and u will be leaving the basis.

8) Divide all of the numbers in the pivotal row by the pivot. This gives

Basic Variables	X	Y	S	T	U	Values
S	$^{19}/_2$	0	1	$^{-5}/_2$	0	250
T	$^{1}/_2$	1	0	$^{1}/_2$	0	50
U	1	0	0	$^{-5}/_3$	$^{2}/_3$	$^{50}/_3$
P	$^{-1}/_2$	0	0	$^{5}/_2$	0	250

9) Now subtract multiples of the pivotal row from each of the other rows so that the pivotal column contains 0s in all rows APART from the pivotal row (which contains a 1). The calculations that are required are indicated to the right of the following table, and tableau 3 is given below that:

Basic Variables	X	Y	S	T	U	Values	
S	$^{19}/_2$	0	1	$^{-5}/_2$	0	250	$R_1-^{19}/_2R_3$
T	$^{1}/_2$	1	0	$^{1}/_2$	0	50	$R_2-^{1}/_2R_3$
U	1	0	0	$^{-5}/_3$	$^{2}/_3$	$^{50}/_3$	
P	$^{-1}/_2$	0	0	$^{5}/_2$	0	250	$R_4+^{1}/_2R_3$

Tableau 3

Basic Variables	X	Y	S	T	U	Values
S	0	0	1	$^{40}/_3$	$^{-19}/_3$	$^{275}/_3$
T	0	1	0	$^{4}/_3$	$^{-1}/_3$	$^{125}/_3$
U	1	0	0	$^{-5}/_3$	$^{2}/_3$	$^{50}/_3$
P	0	0	0	$^{5}/_3$	$^{1}/_3$	$^{775}/_3$

Having constructed tableau 3, look at its bottom row. Notice that there are NO negative entries in this row and we therefore conclude that it is NOT necessary to construct a further simplex tableau.

10) We now deduce the solution to the linear programming problem as follows. Tableau 3 represents the equations:

$$s + \frac{40}{3}t - \frac{19}{3}u = \frac{275}{3}$$

$$y + \frac{4}{3}t - \frac{1}{3}u = \frac{125}{3}$$

$$x - \frac{5}{3}t + \frac{2}{3}u = \frac{50}{3}$$

$$P + \frac{5}{3}t + \frac{1}{3}u = \frac{775}{3}$$

The solution to the problem is found by putting the non-basic variables equal to 0. The values of the other quantities in the problem are then immediately apparent from these equations – note, in particular, that the value of P obtained in this way is its MAXIMUM POSSIBLE VALUE i.e. the optimal solution we are looking for in the linear programming problem.

The solution to the problem is thus:

The maximum value taken by P is $^{775}/_3$ and this value is taken when $t = 0$, $u = 0$, $s = {}^{275}/_3$, $x = {}^{50}/_3$ and $y = {}^{125}/_3$. For our purposes it is mainly the values of x and y that will be important in answering questions as these will directly relate to quantities referenced in the question itself.

Notes

1) Typically, when the solution to a linear programming problem is stated, the quantities we are interested in are:

- the optimum value of the objective function
- the values of the decision variables

When a problem is solved using the simplex method, it is also straightforward to give the values of the slack variables when the objective function is maximised.

2) Remember, no attempt has been made to JUSTIFY the steps above. For the purposes of solving linear programming problems, it suffices to know how to:

- put a problem in standard form
- construct tableau 1
- construct further tableaux as necessary
- check whether it is necessary to construct a further tableau at each stage

- deduce the optimal solution from the final tableau