Linear Programming.

The basic problem of linear programming, determining the optimal value of a linear function subject to linear constraints, arises in a wide variety of situations.

Review of linear geometry.
Generally most of the constraints will take the form of a linear inequality of the form

$$ax + by \leq c \qquad or \qquad ax + by \geq c$$

where x and y are allowed to vary, b and c are constants.

At the beginning of the year we looked at the equations of lines. These took two forms

$$ax + by = c \qquad or \qquad y = mx + c$$

In order to plot a line we need to get two points on that line.
This can be done by

1: choosing two values for x and finding their corresponding values for y.

2: plotting these two points and then

3: drawing the line that goes through these two points.

Examples 1:
Draw the following lines.

1:	x + 2 y = 5	2:	x - y = 2
3:	2 x + 3 y = 6	4:	y = 2 x - 3

Sub-Regions of the plane.
If we replace the equals sign (=) by an inequality sign (≤ ≥) then we are no longer looking at a line but rather a half-plane, i.e. a line splits the plane in two half-planes.

Example 2:
Sketch the half-plane given by $x + 2y \leq 5$

We first of all draw the line given by $x + 2y = 5$
If we let x = 1 we get y = 2.
If we let x = 3 we get y = 1.

Hence the points (1,2) and (3,1) are on the line

The next step is to connect the points by means of **straight** line.

Finally we see whether the point (0,0) satisfies the inequality

$$0 + 2(0) \leq 5$$

Clearly it does, and so we shade in the side of the line that (0,0) is on.

If (0,0) is on the line then just look at a point that is not on the line.

Exercises 3:
Sketch the half planes given by

1. $x \leq 12$
2. $y \leq 8$
3. $x \geq 0$
4. $y \geq 0$
5. $x \leq y$
6. $x + y \leq 5$

Example 4:
A manufacturer produces two products for sale, pencils and pens. Each pencil generates a profit of 10c and each pen generates a profit of 15c. His profit, in pennies, will therefore be

Profit = no. of pencils sold * 10 + no. of pens sold *15

Thus if he sells 8 pencils and 5 pens, his profit amounts to

8*10 + 5*15 = 155c

We can examine different values for his profit in a graphical manner as follows;

Let x = no. of pencils manufactured each minute
y = no. of pens manufactured each minute

or in tabular form

No. of Pencils (x)	No. of Pens (y)	Profit (P)
1	1	
1	2	
2	1	
2	2	
2	3	
3	2	

Clearly as x and/or y get bigger the profit earned per minute gets bigger. This however is not without some limitations. After all, the number of pens or pencils that one can manufacture per minute is not limitless.

Example 4a:
Re-examine the above situation if at most 8 pens and 12 pencils can be manufactured per minute.

Solution
Let x and y be as before and let P denote the profit earned per minute.
Clearly

$$P = 10x + 15y.$$

Our constraints can be expressed as follows

$$x \leq 12 \qquad y \leq 8$$

We also have the following implicit constraints

$$x \geq 0 \qquad y \geq 0$$

One cannot make a negative number of pencils.

<u>We can visualize our problem as follows.</u>

Step 1: Sketch the region defined by the constraints

Step 2: Find the coordinates of the vertices (corners)

<u>Note</u>

The point we choose to maximize our profit must lie within the feasibility region and so must be given by one of the vertices.

Clearly (x,y) = (12,8) will maximize P.

So our maximum profit per minute will be

P = 10*12 + 15*8 = 120 + 120 =240c/minute.

<u>Example 4b:</u>

Consider the above example again with x and y as before. Suppose we add the additional condition that the machines require supervision from an operator and thus at most 16 items a minute can be manufactured.

<u>Solution</u>

Quantity to be maximized is

P = 10 x + 15 y

Constraints are

$$x \geq 0 \qquad y \geq 0$$

$$x \leq 12 \qquad y \leq 8$$

New condition $x + y \leq 16$

Step 1: Find the feasible region.

Step 2: Find the vertices.

Step 3: Evaluate the function to be maximized at each of the vertices.

Examples 5:

1: Maximize $z = 5x + 3y$
subject to

$$x - y \le 2$$
$$2x + y \le 4$$
$$-3x + 2y \le 6$$
$$x \ge 0 \qquad y \ge 0$$

2: Maximize $z = 6x + 4y$
subject to

$$x \ge 1$$
$$y \ge 1$$
$$x + y \le 14$$
$$x + y \ge 7$$

Example 6:

A company manufactures paint and paint thinner. The thinner is mixed equally with the paint. Some customers will just buy the paint but none will buy the thinner alone. Clearly they must manufacture at least as much paint as thinner. Long term contracts with interior decorators demand that they produce at least 20 liters of paint and 5 liters of thinner per day. A common mixing vat is used for both the paint and the thinner and the vat can deal with 70 liters per day. The marketing manager decides that no more than 50 liters of paint could be sold per day. If paint contributes a profit of € 3 per liter and thinner contributes a profit of € 2.50 per liter, what production plan would maximize profits?

Solution:

Let x = amount of paint produced per day (in liters)
y = amount of paint thinner produced per day (in liters)
P = amount of profit per day

We need to maximize

$$P = 3x + 2.5y$$

subject to the constraints

$$x \ge y \qquad x \ge 20 \qquad y \ge 5$$
$$x + y \le 70 \qquad x \le 50$$

Step 1: Find the feasible region

Step 2: Find the vertices of the feasible region.

A	(x,y) = (20,5)
B	(x,y) = (50,5)
C	(x,y) = (20,20)
D	(x,y) = (50,20)
E	(x,y) = (35,35)

<u>Step 3:</u>
Evaluate the object function P = 3x + 2.5y at each of the vertices.

A	(x,y) = (20,5)	P=3(20)+2.5(5) = 72.5
B	(x,y) = (50,5)	P=3(50)+2.5(5) = 162.5
C	(x,y) = (20,20)	P=3(20)+2.5(20)=110
D	(x,y) = (50,20)	P=3(50)+2.5(20)=200
E	(x,y) = (35,35)	P=3(35)+2.5(35)=192.5

Hence the best production plan for this company is to produce 50 liters of paint and 20 liters of thinner a day.

Note:
If it takes the same time to make one liter of paint as it does to make one litter of thinner then in a 7 hour shift one would make paint for five hours and thinner for two hours.

<u>Example 7:</u>
A factory manufactures two products, cement & sand. Cement requires 5kg of raw material to manufacture one container and sand requires 5kg of raw material for one container. Cement requires 8 hours per container to be manufactured and sand require 14 hours per container to be manufactured. In a given week 25 kg of raw material is available and 56 work hours are available. At most 4 containers of cement can be manufactured in a week and at most 5 of sand. *For every 5 containers of sand manufactured there must be at least one of cement.*

If cement generates a profit of € 200 per container and sand generates a profit of € 350 per container, what amounts of cement & sand should be made to maximize profits?

Solution:

Let X = no. of containers of cement made in a week.
Y = no. of containers of sand made in a week.
P = profit made in a week.

We now have to translate the above English into mathematics.

Cement requires 5kg of raw material to manufacture one container and sand requires 5kg of raw material for one container. In a given week 25 kg of raw material is available.

$$5x + 5y \leq 25$$

Cement requires 8 hours per container to be manufactured and sand requires 14 hours per container to be manufactured. In a given week 56 work hours are available.

$$8x + 14y \leq 56$$

At most 4 containers of cement can be manufactured in a week

$$x \leq 4$$

and at most 5 of sand.

$$y \leq 5$$

For every 5 containers of sand manufactured there must be at least one of cement.

$$5x \geq y$$

This last inequality requires some thought.

If y=5 then we need $x \geq 1$ or

$$5x \geq 5 = y$$

Since it is impossible to make a negative amount of cement or sand then we implicitly have that

$$x \geq 0 \qquad y \geq 0$$

If cement generates a profit of € 200 per container and sand generates a profit of € 350 per container then the profit can be given by

P = 200 x + 350 y

In summary we have that we are trying to maximize

P = 200 x + 350 y

subject to the constraints

$$x \geq 0 \qquad y \geq 0$$

$$x \leq 4 \qquad y \leq 5$$

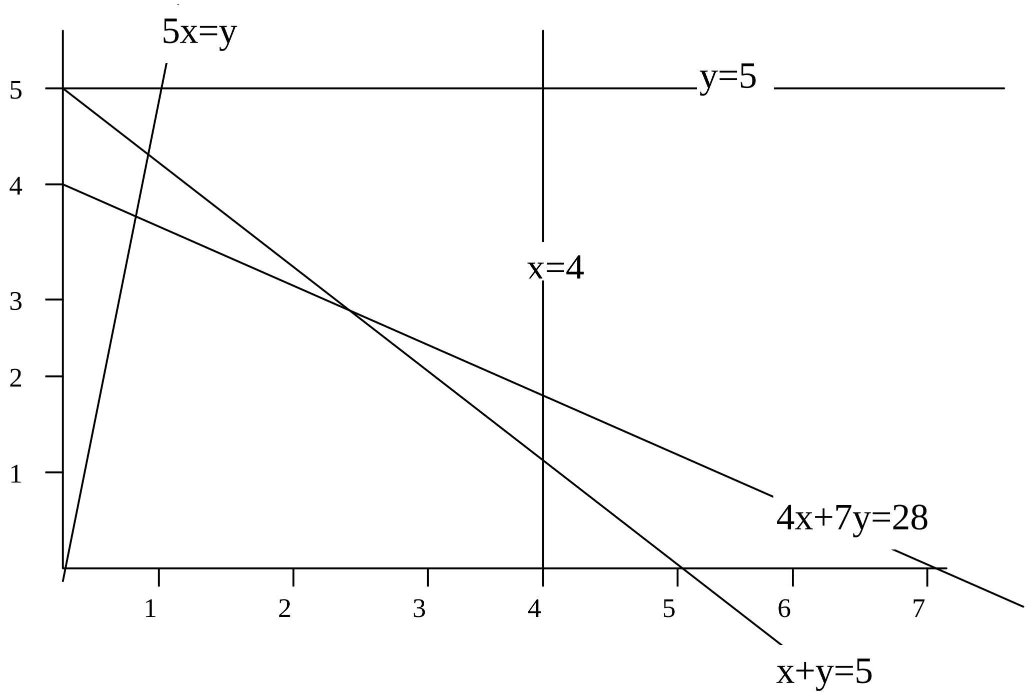
$$5x \geq y$$

$$8x + 14y \leq 56$$

$$5x + 5y \leq 25$$

This gives rise to the feasible region

Step 1:



Exercises 8:

1: Find the optimal values for F = 3 x + 4 y subject to the constraints

$$x \geq 0 \qquad y \geq 0$$

$$x + 2y \leq 6$$

$$4x + 2y \leq 10$$

2: A company sells two products, X at a profit of € 120 and Y at a profit of € 150. At least 5 of each must be made per day. There must be more of X manufactured than Y. No more than 20 items may be made per day.

(a) Set up this problem in the context of linear programming.

(b) Find the production plan which maximizes profit using graphical methods.

(c) If each item takes equal lengths of time to be manufactured, then what proportion of the time is spent manufacturing product X.