Gaussian Elimination

Introduction

Guassian Elimination is a technique used to solve systems of equations using matrices. The equations are first converted into an augmented matrix form whereupon a technique known as Elementary Row Operations can be performed in order to resolve the unknowns in the equations. See Figure 1 below.

$$x + 3y = 5$$
$$2x + 2y = 10$$

$$\left(\begin{array}{cc|c} 1 & 3 & 5 \\ 2 & 2 & 10 \end{array}\right)$$

Figure 1 - Augmented Matrix

The basic goal of Gaussian elimination is to derive a matrix that is in, what is known as, “upper triangular” or “row echelon” format. Put simply this means that the matrix should contain noting but zeroes in the bottom triangular section of the matrix, see Figure 2 below.

$$\left(\begin{array}{cc|c} 1 & 3 & 5 \\ 0 & -4 & 0 \end{array}\right)$$

Figure 2 - Upper Triangular Format Matrix

Augmented Matrix

The Augmented Matrix is a term given to the matrix that represents the system of equations by their coefficients alone. This means that the coefficient of x and the coefficient of y plus the value on the right of the equals sign (the number) make up each row of the matrix. An example of how equations are put into the augmented matrix form is shown in Figure 1. The columns representing the coefficient values are separated from the column representing the numbers values by a straight, vertical line.

Elementary Row Operations

Elementary row operations consist of a combination of any of the following three possibilities:

1. Interchange two rows of the matrix
2. Divide of multiply a row by a non-zero value
3. Add/subtract a multiple of one row to/from another row

Row Echelon Form of a Matrix

The row echelon form of a matrix is rigidly defined by the following points:

1. All rows consisting of 0's occur at the bottom of the matrix.
2. For each non-zero row, the first element must be 1, called the leading one. This is a compulsory property of every row echelon form matrix.
3. Only zeros can appear under each leading one. This is a compulsory property of every row echelon form matrix.

Working with an Example

In our example above:

$$x + 3y = 5$$
$$2x + 2y = 10$$

We converted the equations into an augmented matrix as shown below:

$$\left(\begin{array}{cc|c} 1 & 3 & 5 \\ 2 & 2 & 10 \end{array}\right)$$

In order to put this matrix in row echelon form we require that the bottom row starts with a 0. The only way in which this can happen is if the first element of the bottom row has 2 subtracted from it and this must be achieved by the use of the elementary row operations.

It is logical to see that if we subtract twice the first row from the second row it will result in a zero as the first element of the second row because 2 – 2(1) = 0. Hence our new matrix becomes:

$$\left(\begin{array}{cc|c} 1 & 3 & 5 \\ 0 & -4 & 0 \end{array}\right)$$

Our matrix is now in upper triangular format as the bottom triangular part of the matrix contains only 0's (very easy to achieve in a 2-row matrix example).

Finding the Answer

Once the matrix has been put into row echelon form (upper triangular format). The acquisition of the solution becomes very simple. As we already know the elements of a matrix have a relationship to their positioning within such. For example when we worked with systems of simultaneous linear equations and used matrices to solve the problem, we could map the x and y values from the variable matrix directly onto the values of the answer matrix.

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

Figure 3 - Matrix Mapping

In Figure 3 above we know that x will assume the value 3 and that y will assume the value 4 because the dimensions of the matrices are the same and the order of the variables corresponds to the order of the values.

In the matrix that we just derived from our elementary row operation:

$$\left(\begin{array}{cc|c} 1 & 3 & 5 \\ 0 & -4 & 0 \end{array}\right)$$

We can read this as saying that:

$$\begin{aligned} 1x + 3y &= 5 \\ 0x - 4y &= 0 \end{aligned}$$

Using the last row to our advantage we can conclude that if -4y = 0 (because of 0x not being part of the equation) then y = 0. Back substituting this value for y into the top row we now derive a value for x.

$$\begin{aligned} 1x + 3(0) &= 5 \\ 1x + 0 &= 5 \\ x &= 5 \end{aligned}$$

Now we have both a value for x and also a value for y, meaning that we have solved the system of equations and derived a point of intersection that indicates where the two lines represented by the original equations meet.