Cramer's Rule

Introduction

Cramer's Rule is a method for solving systems of equations that is loosely based on matrices. More aptly Cramer's Rule is based upon a specific area of matrix calculations, namely the determinant of a matrix. The greatest advantage of Cramer's Rule apart from its conciseness is that it allows for the direct calculation of any unknown in a system of equations without the need to solve the entire system.

Previously with simultaneous linear equations and solving these by the standard matrix method it was only possible to solve the entire set of equations at once. Cramer's Rule removes this overhead and speeds up the process of calculating unknowns.

How to do it

Cramer's Rule states that the system of equations must first be converted into matrix form before any work can proceed. Once the coefficients have been correctly transferred into matrix form, the first step can commence. To illustrate clearly how the method works we will adopt an example with which to explain the sequence of steps. Imagine the following linear equations: solve for x and y.

$$3x - 5y = 12$$
$$7x + 2y = 6$$

Convert to matrix form:

$$\begin{pmatrix} 3 & -5 \\ 7 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 12 \\ 6 \end{pmatrix}$$

Step 1: Calculate the determinant of the coefficient matrix (the leftmost matrix). This value will become known as D.

$$\begin{pmatrix} 3 & -5 \\ 7 & 2 \end{pmatrix}$$

$$\mathrm{D} = 3(2) - (-5)(7) = 6 - (-35) = 41$$

Step 2: Substitute the values in the result matrix (i.e. rightmost matrix) for each y coefficient in the leftmost matrix (i.e. substitute 12 for -5 and 6 for 2). Then calculate the determinant of this new matrix. This determinant will be called D_y.

$$\begin{pmatrix} 3 & 12 \\ 7 & 6 \end{pmatrix}$$

$$D_y = 3(6) - 12(7) = 18 - 84 = -66$$

Step 3: Starting with the original matrices again: Substitute the values in the result matrix for each x coefficient in the leftmost matrix (i.e. substitute 12 for 3 and 6 for 7). Then calculate the determinant of this new matrix. This determinant will be called D_x.

$$\begin{pmatrix} 12 & -5 \\ 6 & 2 \end{pmatrix}$$

$$D_x = 12(2) - (-5)(6) = 24 - (-30) = 54$$

Step 4: Calculate the value of x by dividing D_x by D and similarly calculate the value of y by dividing D_y by D. These values for x and y are the solution to the system of equations and in our example above are the coordinates of the point of intersection of the two lines given by the original equations.

$$x = \frac{-66}{41}$$
$$y = \frac{54}{41}$$

Step5: As a sanity check you can back substitute the new values for x and y into the original equations and see if they values satisfy the equalities.

$$3\left(\frac{54}{41}\right) - 5\left(\frac{-66}{41}\right) = 12$$
$$\frac{162}{41} + \frac{330}{41} = 12$$
$$\frac{492}{41} = 12, \textit{true as } 492 \div 41 = 12$$

We can now see that the x and y values calculated are correct as they satisfy the original equations. Hence we have used Cramer's Rule correctly.